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VARIATIONAL MODELS AND METHODS IN SOLID AND FLUID MECHANICS

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For this would be agreed by all: that Nature does nothing in vain nor labours in vain

Olympiodorus, Commentary on Aristotle’s Meteora translated by Ivor Thomas in the Greek Mathematica Works Loeb Classical Library

La nature, dans la production de ses effets, agit toujours par les voies les plus simples

Pierre de Fermat

The CISM course C-1006 ”Variational models and methods in solid and fluid mechanics” was held July 12-16, 2010 in Udine, Italy. There were about forty five participants from different european countries. The papers included in this volume correspond to the content of five mini-courses of 6 hours each which have been delivered during this week.

Variational formulation of the governing equations of solid and fluid mechanics is a classical but a very challenging topic. Variational methods give an efficient and elegant way to formulate and solve mathematical problems that are of interest for scientists and engineers. This formulation allows for an easier justification of the well-posedness of mathematical problems, the study of stability of particular solutions, a simpler implementation of numerical methods. Often, mechanical problems are more naturally posed by means of a variational method. Hamilton’s principle of stationary (or least) action is the conceptual basis of practically all models in physics. The variational formulation is also useful for obtaining simpler approximate asymptotical models as done in the theory of homogeneization. In many problems of mechanics and physics, the functionals being minimized depend on parameters which can be considered as random
variables. Variational structure of such problems always brings considerable simplifications in their study.

In this course, three fundamental aspects of the variational formulation of mechanics will be presented: physical, mathematical and applicative ones.

The first aspect concerns the investigation of the nature of real physical problems with the aim of finding the best variational formulation suitable to those problems. A deep knowledge of the physical problems is needed to determine the Lagrangian of the system and the nature of the variations of its motions which may be considered admissible. Actually one could say that all knowledge which is available about a system is resumed by the choice of:

- a configuration space used to describe mathematically the system
- a set of admissible motions used to describe the different ways in which the system may evolve
- a Lagrangian functional which once minimized supplies evolution equations and boundary conditions

The second aspect is the study of the well-posedness of those mathematical problems which need to be solved in order to draw previsions from the formulated models. It is relatively simple to conjecture properties to be required to the Lagrangian functional in order to be assured the well-posedness of the corresponding evolution system. Much more complex is to get such results of well-posedness studying some evolution equations which are obtained with euristic schemes different from those based on Hamilton’s principle. In fact always, when one needs to study mathematically a set of evolution equations, the first move is to try to put them in a variational form. It is then advisable and wiser to try to use a variational principle at the beginning of the formulation of a mathematical model.

The third aspect is related to the direct application of variational analysis to solve real engineering problems. Variational principles supply very powerful tools for getting qualitative previsions about the behaviour of the studied systems, but also for formulating effective numerical methods to get quantitative previsions.

The following problems have been presented and studied during the course:

- Rayleigh-Hamilton’s Principle for establishing governing equations and boundary conditions for second gradient models for heterogeneous deformable bodies;
• A variational approach to multiphase flow problems and description of diffuse solid-fluid interfaces;
• New variational models of brittle fracture mechanics and some related problems;
• The methods of stochastic calculus of variations and their applications to the homogenization problems and modeling of microstructures and their evolution;
• Dynamical problems in damping generation and control in the situations where the energy initially conferred to a system undergoes a principle of irreversible energy confinement into a small region;

We are extremely grateful to all participants of the course for creating a nice atmosphere for scientific discussions. We would like also to express our thanks to the CISM staff for their assistance in the running of the course.

Francesco dell’Isola, University of Rome "La Sapienza"

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Variational principles are a powerful tool also for formulating field theories

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Abstract Variational principles and calculus of variations have always been an important tool for formulating mathematical models for physical phenomena. Variational methods give an efficient and elegant way to formulate and solve mathematical problems that are of interest for scientists and engineers and are the main tool for the axiomatization of physical theories.

1 Introduction and historical background

1.1 Metrodoron and his followers

The ideas we want to evoke in this lecture are very old and were put forward already in the hellenistic period: for a detailed discussion about this point the reader is referred to the beautiful book by Lucio Russo (2003). In that book it is established that “modern” science actually was born in the hellenistic era, when Metrodoron lived. Metrodoron was a pupil of a famous greek philosopher, Epicurus, and, in our opinion, the following Metrodoron’s sentence is a statement (the first?) belonging to the modern philosophy of science:

«Μέμνεσο ὅτι θνητός ἦν τῇ φύσι καὶ λαβὼν χρόνον ἑωμένον ἀνέβης τοῖς περὶ φύσεως διαλογισμοῖς ἐπὶ τὴν ἀπαιγίαν καὶ τὸν αἰώνα καὶ κατειδέες “τὰ τ’ ἑόντα τ’ ἑοόμενα πρὸ τ’ ἑόντα”».

Metrodoron,

“Always remember that you were born mortal and such is your nature and you were given a limited time: but by means of your reasonings about Nature you could rise to infinity and to eternity
and you indeed contemplate “the things that were, and that were to be, and that had been before”. Metrodoron


This dictum, following Körte, comes from a lost letter or book by Metrodoron (the Epicurean philosopher) addressed to Menestratos who was presumably one of his pupils. The words quoted in bold are a citation from Iliad, I 70 (the translation into English of the sentence in boldface is ours; except for this citation the translation has been taken from Homer by Murray (1924), see the ref. (14)).

In different words, Metrodoron states that by using (the right!) equations you can forecast future behavior of physical systems.

1.2 Why Variational Principles and Calculus of Variation?

In recent time, a lost Archimedes’ book (19) has been rediscovered. Some authors claim that Archimedes seems to have solved, in this book and using a variational principle, the technological problem of finding the optimal shape of a boat. Archimedes seems to have chosen, as optimality criterion, that the vertical position must be a “very” stable configuration (see Rorres (2004)). In the book of Russo (21) it is demonstrated in even a more convincing way that many optimization techniques were well-known in hellenistic science. In particular Russo proves that the problem of the determination of the regular polygon having maximal area has been solved in that period. Thus, the use of a variational principle and optimization methods to solve technological problems is less recent than it is usually believed. In general, variational formulation of the governing equations of solid and fluid mechanics is a classical but very challenging topic. This kind of formulation allows for an easier proof of the well-posedness of mathematical problems, for an easier investigation of the study of stability of particular solutions, and for a simpler implementation of numerical methods. Often (but one who believes in Russo’s vision about the birth of science could say instead “always”), mechanical problems are more naturally posed by means of variational methods. Hamilton’s principle of stationary (or least) Action is the conceptual basis of practically all models in physics. The variational formulation is also useful for obtaining simpler approximate asymptotical models as it is done in the theory of homogenization.

We want simply to state here that the Principle of Virtual Works and the Principle of Least Action have roots much deeper than many scientists believe (see Vailati, 1897). Although many histories of science claim dif-
 differently, most likely the majority of physical theories were first formulated in terms of these Principles, and only subsequently they were reconsidered from other points of view. In our opinion the Principle of Least Action, which supplies a “geometric” version of mechanics was indeed the tool used by the true founders of mechanics (i.e. the scientists of the hellenistic period) to establish it. As argued also by Colonnetti (5) and Netz and Noel (19)) surely also Archimedes and ancient greek scientists were accepting such a point of view.

The epigones of the hellenistic science, who were not able to understand the delicate mathematical arguments used by the first scientists, however could understand the minimality conditions obtained by their “maitres” (i.e. the conditions corresponding to those which we call now Euler-Lagrange equations and boundary conditions) and could grasp the “physical” arguments used to interpret them. This phenomenon is perfectly clear to everyone who is ready to consider carefully -for instance- the evolution of the theory of Euler-Bernoulli Beam (a useful reference about this point is the book of Benvenuto (1981)). Euler postulated a Principle of Least Action for the Elastica, and gets the celebrated equilibrium differential equation and boundary conditions for the equilibrium of beams by using the calculation procedure due to Lagrange (which is the departing idea of Calculus of Variations). Then Navier prepared his lectures for the Ecole Polytechnique and resumed the results obtained by Euler deciding to “spare” to the (engineering) students the difficulties of the calculus of variations. He started directly from the equilibrium equation, obtained by means of an “ad hoc” principle of balance of force and couple, and imposed boundary conditions based on “physical” assumptions. As a consequence, for a long while, generations of engineers believed that the beam equations were to be obtained in this way. Only when numerical simulations became popular, then they (actually, some of them) became aware of variational “principles”. However these principle were proven as theorems starting from “balance postulates” and were considered simply as a mathematical (rather abstruse) tool and not as a fundamental heuristic concept. And this attitude is not changed even when it became clear that every serious advancement of mechanical science has been obtained using variational principles. Indeed the so called “physical sense” (a gift that many claim to posses but which nobody can claim to be able to master or to teach) is not very useful to postulate the right “balance principles” when one is in “terra incognita”. For instance, when Lagrange and Sophie Germain wanted to find the plate equations they needed to employ a variational principle (and they could find the (right!) natural boundary conditions). Again when Cosserat brothers wanted to improve Cauchy Continuum Mechanics they “rediscovered” the right tech-
nique: i.e. the Principle of Least Action. Also Quantum Mechanics has been developed starting from a Variational Principle (see e.g. the references of Feynman (11), Lagrange (15) and Lanczos (16)).

Therefore an important warning is due to young researcher: refrain from trying to extend available models by means of “ad hoc” adaptations of available theories: always look for the right Action functional to be minimized!

1.3 The problem of including dissipation

One useful tool for handling complicated situations is used in Continuum Mechanics by Paul Germain when formulating second gradient theories: the Principle of Virtual Powers. Again, as remarked always in the history of the development of ideas, when this history can be reconstructed, the effective way to be used to proceed is that which starts from a Principle of Least Action, eventually generalized into a Principle of Virtual Powers. For a long time the opponents to Second Gradient Theories argued about its lack of consistency, due to the difficulties they claim to find in “getting” boundary conditions. This is a really odd statement. Indeed variational principles easily produce mathematically correct boundary conditions. So maybe what those opponents want to say is that as they are not so clever as Navier, they do not manage to interpret physically the boundary conditions found via a (correct and meaningful) variational principle. Of course if one refuses to use the Principle of Least Action he can find very difficult the job of determining some set of boundary conditions which are compatible with the (independently postulated!) bulk evolution equations. If instead one accepts the Archimedean (the reader will allow us to dream, without definitive evidence that such was the point of view of Archimedes) approach to mechanics then all these problems of well-posedness of mathematical models completely disappear.

Variational Principles always produce intrinsically well-posed mathematical problems, if the Action functional is well behaving. Of course passing from Lagrangian systems (the evolution of which are governed by a Least Action functional) to non-Lagragian systems (for which such a functional may not exist) may be difficult. This problem is related (but is not completely equivalent) to the problem of modelling dissipative phenomena. It is often stated that dissipation cannot be described by means of a Least Action Principle. This is not exactly true, as it is possible to find some Action functionals for a large class of dissipative systems. However it is true that not every conceived system can be regarded as a Lagragian one. This point is delicate and will be only evocated here. In general a non-Lagragian system can be regarded as Lagragian in two different ways: i) because it
is an “approximation” of a Lagrangian system (see the case of Cattaneo equation for heat propagation), and this approximation leads to “cancel” the lacking part of the “true” Action Functional ii) because the considered system is simply a subsystem of a larger one which is truly Lagrangian. The assumption that variational principle can be used only for non-dissipative systems is contradicted by, e.g., the work presented in this book by Prof. Frankfort (12), where you find variational principles modelling dissipative systems. Indeed it is often stated that a limit of the modelling procedure based on variational principles consists in their impossibility of encompassing “nonconservative” phenomena. We do not believe that this is the case: however in order to avoid to be involved in a problem which is very difficult to treat, when dealing with dissipative systems, we will assume a slightly different point of view, usually attributed to Hamilton and Rayleigh.

2 Finding a mathematical model for natural phenomena

2.1 Principle of Least Action

We want to discuss here about the problem of finding a mathematical model for natural phenomena. We start with an epistemological Principle:

“The Principle of Least Action tells us how to construct a mathematical model to be used for describing the world and for predicting the evolution of the phenomena occurring in it”.

In the following modeling scheme, we give the right heuristic strategy to be used for finding an effective model using the Principle of Least Action. The recipe includes the following ingredients:

1. Establish the right kinematics needed to describe the physical phenomena of interest, i.e. the kinematical descriptors modeling the state of considered physical systems.
2. Establish the set of admissible motions for the system under description, i.e. establish the correct model for the admissible evolution of the system.
3. Employ the “physical intuition” to find the right Action functional to be minimized, i.e. modeling what Nature wants to minimize.

We start by finding the kinematical descriptors, because of the need of modeling the states of the considered system. Then we introduce motion, in such a way we model the evolution of the system to be described. Finally we ask Nature what is the quantity to minimize. Keeping this quantity in mind, we introduce the Action functional. To start with, it is necessary
to focus the attention on a specific class of systems and on phenomena occurring to them. A configuration is the mathematical object used to model the state of considered systems: the set of possible configurations will be denoted by $C$. The motion is the mathematical model describing the evolution of considered systems: it is a $C$-valued function defined on time interval $(t_0, t_f)$; the set of all admissible motions will be denoted by $M$. The Action is a real-valued function, defined on $M$, which models the “preferences” of nature.

Finally, to use the Principle of Least Action one needs three steps further,

4. Find the Euler-Lagrange conditions which are consequence of the postulated Least Action Principle
5. Interpret these condition on a physical ground
6. Determine, in terms of the postulated Action functional, the numerical integration scheme to be used to get the previsions needed to drive, by means of our theory, our experimental, technological or engineering activity.

2.2 The Rayleigh-Hamilton principle

When postulating an extended Rayleigh-Hamilton principle, the point 4 of the previously presented heuristic strategy will be further divided into two steps as follows:

4a. Once the quantities which expend power on the considered velocity fields are known in terms of postulated Action, introduce a suitable definite positive Rayleigh dissipation functional
4b. Equate the first variation of Action functional to the Rayleigh dissipation functional and get the evolution equations (including boundary conditions) which govern the motion of the system

Although in the literature the choice of including a Rayleigh-Hamilton principle in the class of variational principles is sometimes considered inappropriate, we will follow what seems to us the preference of the majority of the authors: therefore we do call “variational” also the strategy which we just described, not limiting the use of this adjective to the models using exclusively the Least Action Principle.

2.3 La Cinématique d’Abord!

According to Metrodoron, mathematical and physical objects are two different concepts. Indeed, equations are necessary for modeling physical systems but they refer to mathematical objects. When one solves the equations formulated in the framework of his model then he has to transform the